On planar biaxial tests for anisotropic nonlinearly elastic solids. A continuum mechanical framework

GERHARD A. HOLZAPFEL
Institute for Biomechanics, Center for Biomedical Engineering, Graz University of Technology, Kronesgasse 5-I, A–8010 Graz, Austria
Department of Solid Mechanics, School of Engineering Sciences, Royal Institute of Technology (KTH), Osquars Backe 1, SE-100 44 Stockholm, Sweden

RAY W. OGDEN
Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, UK

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Abstract: The mechanical testing of anisotropic nonlinearly elastic solids is a topic of considerable and increasing interest. The results of such testing are important, in particular, for the characterization of the material properties and the development of constitutive laws that can be used for predictive purposes. However, the literature on this topic in the context of soft tissue biomechanics, in particular, includes some papers that are misleading since they contain errors and false statements. Claims that planar biaxial testing can fully characterize the three-dimensional anisotropic elastic properties of soft tissues are incorrect. There is therefore a need to clarify the extent to which biaxial testing can be used for determining the elastic properties of these materials. In this paper this is explained on the basis of the equations of finite deformation transversely isotropic elasticity, and general planar anisotropic elasticity. It is shown that it is theoretically impossible to fully characterize the properties of anisotropic elastic materials using such tests unless some assumption is made that enables a suitable subclass of models to be preselected. Moreover, it is shown that certain assumptions underlying the analysis of planar biaxial tests are inconsistent with the classical linear theory of orthotropic elasticity. Possible sets of independent tests required for full material characterization are then enumerated.

Key Words: biaxial testing, anisotropic material, elastic material, soft tissue mechanics, constitutive modeling, finite elasticity

1. INTRODUCTION

In some recent papers there have appeared statements such as “Since biological tissues are generally considered incompressible, planar biaxial testing allows for a two-dimensional stress-state that can be used to characterize fully their mechanical properties.” (see the Ab-
strat of Sacks [1], perpetuated in the Abstract of the review article of Sacks and Sun [2] with similar wording). Although this statement is true in the case of incompressible isotropic materials it is false for anisotropic materials such as tissues, and, therefore, has the potential to mislead some researchers on the value of planar biaxial testing. Equally, the recent paper by Wang et al. [3] with the title “Three-dimensional mechanical properties of porcine coronary arteries: a validated two-layer model” does not, contrary to its claim, provide three-dimensional mechanical properties. There is therefore a need to clarify the value of biaxial testing, and what tests are required for determination of the three-dimensional material properties.

The researchers cited above use the results of planar biaxial testing or extension–inflation testing, and claim that they are characterizing the three-dimensional mechanical properties of the material. This is an invalid claim. Indeed, we show in this paper that it is theoretically impossible to determine the three-dimensional properties of anisotropic elastic materials from biaxial tests unless some assumption is made that enables a suitable subclass of models to be preselected. However, the process of preselection presupposes some prior information about the material characteristics that may or may not be justified. An example where this point is not appreciated is in the paper by Sun et al. [4]. They claim that it is the model form of the constitutive law that is being determined. This is not the case since they preselect a standard Fung-type model with a limited number of constants, which are determined by a data-fitting exercise. Typically in the literature, a two-dimensional Fung model is used, sometimes without an in-plane shear (see, for example, the recent study by Pandit et al. [5]). Moreover, the recent use by Wang et al. [3] of a three-dimensional Fung-type model does not provide a model that can, as stated in their paper, “...serve as a basis for formulation of various boundary-value problems...” except for limited problems that do not involve shear. Indeed, the model they adopt cannot be used to solve any realistic boundary-value problem involving shear since it does not include shear terms.

It is worth emphasizing the very important point that two-dimensional or membrane-type models cannot describe three-dimensional material properties. In particular, features such as residual stresses, through-thickness stress distributions, and torsional deformations cannot be captured.

For our purposes it suffices to focus initially on the simplest representation of material anisotropy, namely transverse isotropy. The aim of this paper is to use this representation to explain the limitations of biaxial testing, to point out some errors in the literature and to identify the needs for the mechanical testing of anisotropic solids capable of large elastic deformations, including soft biological tissues. We emphasize that the experiments and the claims referred to herein are based on modeling the material as hyperelastic. Thus, our development is also based on this theory. To the extent that the material can be considered to be elastic, features such as time dependence, multi-phasic material, etc., are strictly not relevant to the discussion. However, we note that if one wants to account for these more general features then the required range of experiments would be more extensive than those discussed in the paper. The hyperelastic framework is therefore sufficient to highlight the issues.

We begin, in Section 2, by recalling the equations associated with the planar biaxial testing of isotropic elastic solids that were originally developed in the context of rubber
elasticity. For isotropic materials, planar biaxial tests are enough to fully characterize the materials properties. In Section 2.2 we derive the corresponding equations for a general transversely isotropic elastic solid such as is often used for the description of fibre-reinforced materials, including some soft biological tissues. Here we show why planar biaxial testing is insufficient for complete characterization of such materials. In order to emphasize this point, in Section 3, we use a general formulation of planar anisotropy to establish this conclusion. This general theory is then applied in the case of the two-dimensional Fung model alluded to above. Restrictions on the constants of this model for compatibility with the classical linear theory of orthotropic elasticity are then noted. We proceed by suggesting a minimum portfolio of tests that could serve the purpose of fully characterizing the three-dimensional material properties. In Section 4 we refer briefly to the corresponding situation for orthotropic materials, and this is followed by a concluding discussion.

2. PLANAR BIAXIAL DEFORMATIONS

2.1. Isotropic Materials

Materials such as rubber can, to some extent, be treated as isotropic elastic. The properties of an isotropic elastic solid are characterized in terms of a strain-energy function, denoted $W$ and defined per unit reference volume. For an incompressible material $W$ depends on just two independent measures of deformation. Typically, these are taken as the two principal invariants, $I_1$ and $I_2$, of the right Cauchy–Green tensor, $C = F^T F$ (or, equivalently, of the left Cauchy–Green tensor, $B = F F^T$), where $F$ is the deformation gradient and $(\bullet)^T$ denotes the transpose (of a second-order tensor $(\bullet)$). These are defined by

$$I_1 = \text{tr} C = \text{tr} B, \quad I_2 = \frac{1}{2} \left[ I_1^2 - \text{tr} (C^2) \right] = \frac{1}{2} \left[ I_1^2 - \text{tr} (B^2) \right],$$

where $\text{tr} (\bullet)$ denotes the trace (of a second-order tensor $(\bullet)$). (Recall that the third principal invariant $I_3 = \det C = \det B$ is identically equal to 1 for an incompressible material.) Thus, we write $W = W(I_1, I_2)$. The associated Cauchy stress tensor is given by the standard representation

$$\sigma = -p I + 2W_1 B + 2W_2 (I_1 B - B^2),$$

where $W_i = \partial W / \partial I_i, i = 1, 2$, $p$ is the Lagrange multiplier associated with the incompressibility constraint and $I$ is the identity tensor. We recall that for a thin sheet we have the approximation that the through-thickness (principal) Cauchy stress $\sigma_3 = 0$. It follows from (2) by specializing to the principal axes of $B$ and eliminating $p$ that the corresponding in-plane principal Cauchy stresses $\sigma_1$ and $\sigma_2$ for pure homogeneous planar biaxial deformation of a thin sheet are written simply as

$$\begin{align*}
\sigma_1 &= 2 \left( \lambda_1^2 - \lambda_1^{-2} \lambda_2^{-2} \right) \left( W_1 + \lambda_2^2 W_2 \right), \\
\sigma_2 &= 2 \left( \lambda_2^2 - \lambda_1^{-2} \lambda_2^{-2} \right) \left( W_1 + \lambda_1^2 W_2 \right),
\end{align*}$$

(3)
where $\lambda_1$ and $\lambda_2$ are the in-plane principal stretches (see, for example, [6], Equation (6.5) therein). Note that the stretch $\lambda_3$ is given by the incompressibility condition $I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2 = 1$ and that, in terms of $\lambda_1$ and $\lambda_2$, we have $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2}$ and $I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2 \lambda_2^2$.

Within the framework of Equations (3), Rivlin and Saunders [7] were the first to conduct systematic series of biaxial experimental tests for rubber materials. They were able to fix either $I_1$ or $I_2$ while varying $I_2$ or $I_1$, respectively.

An alternative and equivalent pair of equations, with $W = \hat{W}(\lambda_1, \lambda_2)$ treated directly as a symmetric function of $\lambda_1$ and $\lambda_2$, is simply (see, for example, [8, 9])

$$\sigma_1 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{4}$$

Equations (4) form a simple basis for the planar biaxial evaluation of isotropic elastic materials in which one of the stretches ($\lambda_1$ or $\lambda_2$) is held fixed as the other is varied. A systematic series of experiments of this kind was performed for vulcanized rubber sheets by Jones and Treloar [10] (see also the account in [11], Section 4.2.1). For soft biological tissue it was Lanir and Fung [12] who performed the first biaxial tests, in particular on rabbit skin.

It is important to emphasize that, whether Equations (3) or (4) are used, in either case the energy function depends on only two deformation variables, which can be varied independently in planar biaxial tests, and two components of stress, which can be measured along with the deformation (strictly it is the applied forces that are measured and values of the stress components are deduced therefrom). Thus, data from planar biaxial tests are sufficient to fully characterize the material properties, and consequently, based on experimentally measurable quantities, to suggest specific mathematical forms of $W$, for a wide range of deformations (see also [7]). As we show in Section 2.2, this is not the case for anisotropic solids even in the simplest representation of material anisotropy, i.e. transverse isotropy. An alternative and effective way of achieving biaxial deformation is through extension–inflation tests on circular cylindrical tubes with closed ends (see, for example, [13], within the context of rubber elasticity).

In the following section we consider the modifications of the above isotropic theory required in the case of a transversely isotropic material.

### 2.2. Transversely Isotropic Materials

Transversely isotropic elastic solids are characterized by the existence of a single preferred direction, which here we denote by the unit vector $\mathbf{M}$ in the reference configuration, but otherwise is isotropic. (For the present purposes, it suffices to assume that the reference configuration is stress free, although this is not the case in general.) This introduces two invariants, additional to the $I_1$ and $I_2$ defined in (1), that depend on $\mathbf{M}$, or, equivalently, on $\mathbf{m} = F \mathbf{M}$, i.e. the image of $\mathbf{M}$ in the deformed configuration. These additional invariants are denoted $I_4$ and $I_5$ and defined by

$$I_4 = \mathbf{M} \cdot C \mathbf{M} = \mathbf{m}^2, \quad I_5 = \mathbf{M} \cdot C^2 \mathbf{M} = \mathbf{m} \cdot B \mathbf{m}. \tag{5}$$
The associated strain-energy function then depends on four independent invariants: \( W = W(I_1, I_2, I_4, I_5) \), and, for three dimensions, the Cauchy stress tensor is given by

\[
\sigma = -p I + 2 W_1 B + 2 W_2 (I_1 B - B^3) + 2 W_4 m \otimes m + 2 W_5 (m \otimes Bm + Bm \otimes m),
\]

where \( W_i = \partial W / \partial I_i, \ i = 1, 2, 4, 5 \), and the tensor product \( u \otimes v \) of vectors \( u \) and \( v \) is defined by \( (u \otimes v)_{ij} = u_i v_j \). For general background on transversely isotropic constitutive laws we refer to, for example, [14, 15, 16].

Let us now apply this framework to planar biaxial deformations (in the \( (1, 2) \) plane in a Cartesian coordinate system), for which the deformation gradient can be written, in matrix form, as

\[
[F] = \begin{bmatrix}
F_{11} & F_{12} & 0 \\
F_{21} & F_{22} & 0 \\
0 & 0 & F_{33}
\end{bmatrix},
\]

where \( F_{33} = \lambda_3 \) is the stretch normal to the plane, and we have introduced square brackets [\( \bullet \)] for a matrix. Since the material is incompressible,

\[
(F_{11} F_{22} - F_{12}^2) \lambda_3 = 1.
\]

Correspondingly, \( B \) has the matrix form

\[
[B] = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & B_{33}
\end{bmatrix},
\]

with \( B_{11} = F_{11}^2 + F_{12}^2, B_{12} = F_{11} F_{21} + F_{22} F_{12}, B_{22} = F_{21}^2 + F_{22}^2 \) and \( B_{33} = F_{33}^2 = \lambda_3^2 \).

We now take the preferred direction \( M \) to lie in the considered \((1, 2)\) plane. In matrix form this is written as \([M] = [\cos \alpha, \sin \alpha, 0]^T\) and its spatial counterpart \([m] = [F][M]\) as \([m] = [m_1, m_2, 0]^T\), with

\[
m_1 = F_{11} \cos \alpha + F_{12} \sin \alpha, \quad m_2 = F_{21} \cos \alpha + F_{22} \sin \alpha,
\]

where \( \alpha \) denotes the angle that the preferred direction \( M \) makes with the \( 1 \) direction, measured in the counterclockwise sense. In terms of spatial components the invariants (1) and (5) then specialize to

\[
I_1 = B_{11} + B_{22} + B_{33}, \quad I_2 = B_{11} B_{22} + B_{11} B_{33} + B_{22} B_{33} - B_{12}^2,
\]

\[
I_4 = m_1^2 + m_2^2, \quad I_5 = B_{11} m_1^2 + 2 B_{12} m_1 m_2 + B_{22} m_2^2.
\]

According to (6) the non-zero components of \( \sigma \) may therefore be written as
\[
\sigma_{11} = -p + 2W_1B_{11} + 2W_2 \left[ B_{11}(B_{22} + B_{33}) - B_{12}^2 \right] \\
+ 2W_4m_1^2 + 4W_5m_1(B_{11}m_1 + B_{12}m_2), \\
(13)
\]
\[
\sigma_{22} = -p + 2W_1B_{22} + 2W_2 \left[ B_{22}(B_{33} + B_{11}) - B_{12}^2 \right] \\
+ 2W_4m_2^2 + 4W_5m_2(B_{12}m_1 + B_{22}m_2), \\
(14)
\]
\[
\sigma_{12} = 2W_1B_{12} + 2W_2B_{12}B_{33} \\
+ 2W_4m_1m_2 + 2W_5 \left[ m_1m_2(B_{11} + B_{22}) + B_{12}(m_1^2 + m_2^2) \right], \\
(15)
\]
\[
\sigma_{33} = -p + 2W_1B_{33} + 2W_2B_{33}(B_{11} + B_{22}). \\
(16)
\]

Now, the thin sheet or “membrane” approximation allows us to set \( \sigma_{33} = 0 \) and hence (16) can be used to eliminate \( p \) from (13) and (14), and Equations (13)–(16) reduce to the three equations

\[
\sigma_{11} = 2(B_{11} - B_{33})W_1 + 2 \left[ (B_{11} - B_{33})B_{22} - B_{12}^2 \right] W_2 \\
+ 2m_1^2W_4 + 4m_1(B_{11}m_1 + B_{12}m_2)W_5, \\
(17)
\]
\[
\sigma_{22} = 2(B_{22} - B_{33})W_1 + 2 \left[ (B_{22} - B_{33})B_{11} - B_{12}^2 \right] W_2 \\
+ 2m_2^2W_4 + 4m_2(B_{12}m_1 + B_{22}m_2)W_5, \\
(18)
\]
\[
\sigma_{12} = 2B_{12}W_1 + 2B_{12}B_{33}W_2 \\
+ 2m_1m_2W_4 + 2[m_1m_2(B_{11} + B_{22}) + B_{12}(m_1^2 + m_2^2)]W_5. \\
(19)
\]

Note that for an isotropic material referred to the principal axes of \( \mathbf{B} \), so that \( B_{12} = \sigma_{12} = 0 \), Equations (3) are recovered with \( \sigma_{11} = \sigma_1 \), \( \sigma_{22} = \sigma_2 \). It is important to emphasize that these equations are two-dimensional specializations within the framework of a three-dimensional theory and should be distinguished from equations based on a fundamentally two-dimensional theory. In a two-dimensional theory a significant part of the three-dimensional constitutive law is missing.

A key point to note is that Equations (17)–(19) involve four independent constitutive functions, \( W_1, W_2, W_4, W_5 \). However, for the considered planar biaxial deformation only three independent components of deformation are included in these equations, namely \( B_{11}, B_{22}, B_{12} \), whatever the precise form of the in-plane deformation may be. Moreover, there are only three components of stress. From these equations it is impossible to determine the four constitutive functions, i.e. it is theoretically impossible to fully characterize the elastic properties of transversely isotropic materials on the basis of planar biaxial tests alone. Even though this point has been noted previously in the literature by Humphrey and Yin [17], it has not been heeded in recent work. Thus, contrary to claims in the literature, it is impossible to determine the three-dimensional material properties from two-dimensional planar experiments. At the very least one additional independent (three-dimensional) test is required.
In some works it is assumed a priori that one or more of the invariants is absent from the strain-energy function. The situation is then somewhat different since there are sufficient equations to enable, at least in principle, the reduced number of constitutive functions to be determined for a suitable range of deformations. A case in point is the special form of $W$ that depends only on $I_1$ and $I_4$, as discussed by Humphrey [11], Section 10.3.3, for example. Then, biaxial tests can be used to determine the functions $W_1$ and $W_4$ and hence to construct an associated form of $W$.

It is worth noting that if there is no shear then $[F]$ is diagonal, of the form diag[$\lambda_1$, $\lambda_2$, $\lambda_3$] with $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$. Equations (17)–(19) then simplify to

$$
\sigma_{11} = 2(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2})(W_1 + \lambda_2^2 W_2) + 2m_1^2 W_4 + 4\lambda_1^2 m_1^2 W_5, \\
\sigma_{22} = 2(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2})(W_1 + \lambda_1^2 W_2) + 2m_2^2 W_4 + 4\lambda_2^2 m_2^2 W_5, \\
\sigma_{12} = 2m_1 m_2 \left[ W_4 + (\lambda_1^2 + \lambda_2^2) W_5 \right],
$$

with $m_1 = \lambda_1 \cos \alpha$, $m_2 = \lambda_2 \sin \alpha$ deduced from (10), and the invariants (11) and (12) simplify to

$$
I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2}, \quad I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2 \lambda_2^2, \\
I_4 = \lambda_1^4 \cos^2 \alpha + \lambda_2^2 \sin^2 \alpha, \quad I_5 = \lambda_1^4 \cos^2 \alpha + \lambda_2^4 \sin^2 \alpha.
$$

This is the situation for standard planar biaxial tests in the absence of shear, but it is important to observe that if the preferred (fibre) direction is not along one of the coordinate axes then a shear stress is required in order to maintain the (pure homogeneous) deformation. Note that the shear stress vanishes only if $\alpha$ is 0 or $\pi/2$, or if the material is isotropic.

For this special case (with no shear) we may treat $W$ as a function of only two independent stretches, $\lambda_1$ and $\lambda_2$: thus, $W = \tilde{W}(\lambda_1, \lambda_2)$. Note that $W$ also depends implicitly on the angle $\alpha$, which can be regarded as a material constant. Then, Equations (20) and (21) can be written simply as

$$
\sigma_{11} = \lambda_1 \frac{\partial \tilde{W}}{\partial \lambda_1}, \quad \sigma_{22} = \lambda_2 \frac{\partial \tilde{W}}{\partial \lambda_2},
$$

which are identical to the corresponding equations (4) in the isotropic situation ($\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$) except that here $\tilde{W}$ is not a symmetric function of $\lambda_1$ and $\lambda_2$, unlike the situation for isotropy. In fact, Equations (25) also hold if $\alpha$ is not equal to 0 or $\pi/2$, but then $\sigma_{11}$ and $\sigma_{22}$ are not principal stresses and there is no corresponding simplified equation for $\sigma_{12}$ (see the discussion in [18]).

In the following section we provide an alternative and more general treatment of the above analysis in terms of the components of the commonly used Green–Lagrange strain tensor and second Piola–Kirchhoff stress tensor without the need to assume transverse isotropy.
3. IN-PLANE RESPONSE OF AN ANISOTROPIC MATERIAL

We emphasize that the deformation variables used for the formulation of hyperelasticity do not have an influence on the arguments presented. Nevertheless, it is instructive to consider a commonly used three-dimensional formulation for the strain-energy function $W$ for an incompressible material, regarded as a function of the Green–Lagrange strain tensor $E = (C - I)/2$. Then, the second Piola–Kirchhoff stress $S$ is given by

$$ S = -pC^{-1} + \frac{\partial W}{\partial E}, \quad (26) $$

subject to the constraint

$$ \det C \equiv \det(I + 2E) = 1. \quad (27) $$

For planar biaxial deformations, as considered in Section 2.2, we have the components

$$ [C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}, \quad [C]^{-1} = \begin{bmatrix} C_{22}C_{33} & -C_{12}C_{33} & 0 \\ -C_{12}C_{33} & C_{11}C_{33} & 0 \\ 0 & 0 & C_{33}^{-1} \end{bmatrix}, \quad (28) $$

and incompressibility in terms of the components of $C$ and $E$, respectively, gives

$$ C_{33} = \left( C_{11}C_{22} - C_{12}^2 \right)^{-1}, $$

$$ E_{33} = \left\{ \frac{1 + 2(E_{11} + E_{22}) + 4 \left( E_{11}E_{22} - E_{12}^2 \right)}{\left( E_{11}E_{22} - E_{12}^2 \right)} \right\}^{1/2} = \left( 1 + 2E_{11} + 2E_{22} \right)^{1/2} / 2. \quad (29) $$

Under conditions of plane stress (with $S_{13} = S_{23} = S_{33} = 0$) for the considered planar biaxial state $W$ is dependent only on $E_{11}, E_{22}, E_{12}, E_{33}$, so that

$$ W(E) = W(E_{11}, E_{22}, E_{12}, E_{33}). \quad (30) $$

From (26) and (28) we then have

$$ S_{11} = -pC_{22}C_{33} + \frac{\partial W}{\partial E_{11}}, \quad S_{22} = -pC_{11}C_{33} + \frac{\partial W}{\partial E_{22}}, \quad S_{12} = pC_{12}C_{33} + \frac{\partial W}{\partial E_{12}}, \quad S_{33} = -pC_{33}^{-1} + \frac{\partial W}{\partial E_{33}} = 0. \quad (31) \quad (32) $$

But, because of the incompressibility condition (29), only three of the four components $E_{11}, E_{22}, E_{12}, E_{33}$ are independent. We therefore introduce the reduced energy function $W$ defined by
\[
\hat{W}(E_{11}, E_{22}, E_{12}) = W(E_{11}, E_{22}, E_{12}, E_{33}),
\]  
(33)

in the right-hand side of which \(E_{33}\) is a function of \((E_{11}, E_{22}, E_{12})\) given by (29).

This leads to

\[
S_{11} = \frac{\partial \hat{W}}{\partial E_{11}}, \quad S_{22} = \frac{\partial \hat{W}}{\partial E_{22}}, \quad S_{12} = \frac{\partial \hat{W}}{\partial E_{12}} - C_{12}C_{33}^2 \frac{\partial W}{\partial E_{33}}.
\]  
(34)

Since, from (29), \(\partial E_{33}/\partial E_{11} = \partial C_{33}/\partial C_{11} = -C_{33}^2C_{22}\), the first of (34) follows by using the chain rule for partial derivatives, (31)_1 and (32)_2 via the calculation

\[
\frac{\partial \hat{W}}{\partial E_{11}} = \frac{\partial W}{\partial E_{11}} + \frac{\partial W}{\partial E_{33}} \frac{\partial E_{33}}{\partial E_{11}} = S_{11} + pC_{22}C_{33} + pC_{33}^{-1}(-C_{33}^2C_{22}) = S_{11}.
\]  
(35)

Similarly for \(S_{22}\). For \(S_{12}\), from (29) we have \(\partial E_{33}/\partial E_{12} = 2C_{33}^2C_{12}\), and using (32), we find that

\[
\frac{\partial \hat{W}}{\partial E_{12}} = \frac{\partial W}{\partial E_{12}} + \frac{\partial W}{\partial E_{33}} \frac{\partial E_{33}}{\partial E_{12}} = S_{12} + C_{12}C_{33}^2 \frac{\partial W}{\partial E_{33}}.
\]  
(36)

We note in passing that the corresponding calculation in Section 2.2 in [19] is invalid since it confuses two- and three-dimensional formulations and concludes incorrectly that \(\partial W/\partial E_{33} = 0\).

By contrast, if one starts with a two-dimensional theory then \(\hat{W}\) depends only on \(E_{11}, E_{12}, E_{22}\), and the corresponding two-dimensional stresses are given by

\[
S_{11} = \frac{\partial \hat{W}}{\partial E_{11}}, \quad S_{22} = \frac{\partial \hat{W}}{\partial E_{22}}, \quad S_{12} = \frac{\partial \hat{W}}{\partial E_{12}}.
\]  
(37)

Planar deformations then enable one to determine the form of \(\hat{W}\) as a function of \(E_{11}, E_{22}, E_{12}\), but this says nothing about how \(W\) should depend on \(E_{33}\). Consequently, this does not permit the determination of the three-dimensional material properties. In fact there are infinitely many three-dimensional forms of \(W\) that can lead to a given form of \(\hat{W}\), and (37) is not able to discriminate between these. Thus, there is a significant difference between the planar specialization of a three-dimensional strain-energy function and an \textit{a priori} two-dimensional strain-energy function.

The above analysis assumes that the material is incompressible. For a compressible material we have \(S_{33} = \partial W/\partial E_{33}\), and since \(S_{33} = 0\), in contrast to the incompressible case, we have \(\partial W/\partial E_{33} = 0\). This equation determines \(E_{33}\) implicitly in terms of \(E_{11}, E_{22}\) and \(E_{12}\), and hence replaces the incompressibility condition for this case. The function \(\hat{W}\) can then be defined as before, via Equation (33), and again leads to Equations (37). This requires, as in the incompressible case, that the dependence of \(W\) on \(E_{33}\) is known so that \(\hat{W}\) can be found. We emphasize that determination of \(\hat{W}\) on the basis of (37) does not determine the three-dimensional properties of the material since the dependence of \(W\) on \(E_{33}\) is not provided thereby, i.e. the two-dimensional \(\hat{W}\) contains no three-dimensional information.
Moreover it is important to note that a two-dimensional form of \( W \) does not distinguish between compressible and incompressible materials.

**Application to a Fung model.** Several models in the literature are based on a standard exponential model of Fung. In the present context the planar Fung strain-energy function

\[
W = \frac{1}{2}c(e^Q - 1)
\]

has been used, in particular in [1, 2], where

\[
Q = A_1 E_{11}^2 + A_2 E_{22}^2 + 2A_3 E_{11} E_{22} + A_4 E_{12}^2 + 2A_5 E_{12} E_{11} + 2A_6 E_{12} E_{22},
\]

and \( c \neq 0 \) and \( A_1, \ldots, A_6 \) are material constants. This is a specific example of the two-dimensional \( \tilde{W} \) considered above. According to (37) the in-plane components of \( S \) are

\[
S_{11} = c e^Q (A_1 E_{11} + A_3 E_{22} + A_5 E_{12}),
\]

\[
S_{22} = c e^Q (A_2 E_{22} + A_3 E_{11} + A_6 E_{12}),
\]

\[
S_{12} = c e^Q (A_4 E_{12} + A_5 E_{11} + A_6 E_{22}).
\]

These equations apply, in particular, to an orthotropic material for which two of the axes of orthotropy lie in the considered \((1, 2)\) plane. They apply for any in-plane choice of rectangular Cartesian coordinate system. When the axes of orthotropy are chosen to coincide with the Cartesian axes these stress components are consistent with the classical linear theory of orthotropic elasticity if and only if \( A_5 = A_6 = 0 \). In the linear specialization of Equations (40)–(42) the material constants are related to the classical constants, which for convenience we summarize in the Appendix, and denote by \( c_1 = c A_1, c_2 = c A_2, c_3 = c A_3, c_4 = c A_4, \) with \( A_5 = A_6 = 0 \). We note, however, that in [1, 2] non-zero values of the parameters \( A_5 \) and \( A_6 \) were in some cases obtained in fitting the model to experimental data, and in other cases one or both were arbitrarily set to zero. This suggests that the material model considered therein is not orthotropic. Moreover, since (39) is a two-dimensional model it is fundamentally incapable of characterizing three-dimensional properties, as discussed above.

### 3.1. Other Types of Tests

Several multi-axial tests other than the planar biaxial configuration are in common use and can provide appropriate data towards the material characterization. These include: inflation and extension tests on cylindrical segments of arteries and arterial layers, which provide data equivalent to those from planar biaxial tests; see, for example, the investigations by Schulze-Bauer et al. [20, 21] on human iliac and femoral arteries. The tissues in these papers, however, are considered to be orthotropic rather than transversely isotropic. Another possible test involves the simultaneous extension, inflation, and torsion experiments on canine aortas and common carotid arteries: see, for example, [22]. We also mention triaxial shear tests on soft biological tissue (see [23, 24] for the shear characteristics of passive animal myocardium.
and discussion of this mode of deformation), and finally equibiaxial stretching tests (see, for example, the stress-stretch data on epicardium in [25]).

It is worth remarking here that in Humprey et al. [26] experimental guidelines were carefully laid down. In particular, they noted that if it is desired to determine the response functions of the material one should not use specimens for which the fibre angle varies through the thickness, that in-plane shears should be avoided and that the orientation of the fibre family should be at either 0 or 90 degrees to the testing direction. Indeed, in their experiments Humphrey and co-workers have always tried to minimize shear and to avoid inferring shear information from planar biaxial tests because of the inability to impose and control shear stresses. In [27] it was observed that shearing strains are usually much smaller than the extensional strains. They cannot be controlled directly and were therefore treated as negligible. See also [28]. Sacks [1] in fact remarks that biaxial studies of soft tissues are limited because of their inability to include the effects of in-plane shear.

### 3.2. Test Protocols for Full Characterization of Transversely Isotropic Materials

We have emphasized in Section 2.2 that in order to fully characterize the elastic response of an incompressible transversely isotropic material four separate and independent constitutive functions, \( W_1, W_2, W_4, W_5 \), must be determined. This requires the independent control of four separate components of strain and measurement of four associated components of stress (here we mean the measurement of the forces and cross-sectional areas from which the stresses are calculated), or control of the stress components (strictly the forces) and measurement of the resulting strains, or a combination of control of an appropriate number of stress and strain components and measurement of the others. We have also pointed out that planar biaxial tests, even with in-plane shear cannot provide such information. In such tests at most three components of stress (strain) can be controlled, i.e. planar biaxial tests can provide at most three connections between the four constitutive functions. These connections can, of course, be very valuable and go a long way towards the material characterization. However, as mentioned earlier, there does not appear to be available a biaxial testing device that allows independent control and measurement of three components of stress or strain.

The situation is different in the case of extension and inflation tests on tubular specimens when combined with torsion. These tests provide information equivalent to that for biaxial tests with in-plane simple shear since torsion is locally a simple shear in the material plane for a thin-walled tube. A computer-controlled system for performing simultaneous extension, inflation and torsion experiments on cylindrical blood vessels was first developed by Humphrey et al. [22], while Deng et al. [29] designed another triaxial torsion machine and carried out the first systematic study aimed at identifying the shear modulus of rat arteries at various blood pressures and axial stretches. More recently, a similar device has been used by Lu et al. [30] to determine the torsional properties of porcine coronary arteries at different blood pressures and axial stretches, with account taken of the contributions of the intima-media and adventitia, and by Yang et al. [31] in respect of the shear modulus of the rat esophagus. We also mention that a device capable of simple shear tests (at high frequency) has been used by Arbogast et al. [32], while simple shear devices have also been developed by Dokos et al. [23]. In the latter two cases the shear is not combined with biaxial stretching. It should be noted here that care should be taken in using the terminology “the shear
modulus” since for an anisotropic material there is more than one shear modulus in the linear theory.

To obtain a fourth connection between the constitutive functions an out-of-plane (three-dimensional) test is required. This could be, for example, a through-thickness shear test, either on a planar specimen or via an axial or azimuthal shear of a tubular specimen. The collection of such shear data, in particular, would appear to be of paramount importance since there is very little such data available.

Thus, to summarize the above discussion, the minimum (ideal) test portfolio that allows full characterization of the properties of transversely isotropic materials consists of either

- planar biaxial tests with an in-plane shear, and separate through-thickness shear tests, or
- extension and inflation tests on a tubular specimen combined with torsion, and separate through thickness shear tests (axial shear, azimuthal shear or simple shear of a patch cut from the tube).

In each case separate control of four independent strain components is required.

While we await the development of systems that can implement these general protocols or their equivalents, the more data that can be collected the better able we are to understand the range of material response, and hence to quantify the mechanical environment in which cells and matrix function in health, disease or injury. Thus, data from simple tension and equibiaxial tension tests, for example, can contribute alongside standard biaxial and extension–inflation tests (with or without shear), not least in evaluating the type and degree of anisotropy. Such data will remain valuable if and when suitable general protocols are indeed implemented as a means of validating the models constructed on the basis of data collected from the general protocols.

All these considerations must be set against the background that the properties of soft tissues are highly variable. For example, in general they depend significantly on topography, and it is, therefore, important to collect data from as wide a range of specimens and locations as possible in order to account to some degree for statistical variation.

4. DISCUSSION

While a transversely isotropic material requires four invariants for its description, an incompressible orthotropic material requires seven invariants in the general case. Even for the simplest orthotropic model at least three invariants are required and the Cauchy stress therefore depends on three constitutive functions. In this context the three invariants $I_1, I_4, I_6$ are appropriate, where $I_1$ and $I_4$ are as defined here and $I_6$ is the analogue of $I_4$ for a second preferred direction. Thus, even in this simplest scenario, if it can be justified that three invariants are sufficient to describe the material response, biaxial tests of the type described here (those in which only two strain components are varied independently) can never be enough to fully determine those functions, and on that basis, therefore, constitutive modeling cannot determine the multi-axial behavior but only limited approximations to it.
For a three-dimensional material model involving the three invariants $I_1, I_4, I_6$ see, for example, the arterial wall model in [33], which has (a maximum number of) four material constants. Note that the three-dimensional version of the Fung model has ten material constants, and, as discussed in [33], care must be taken to select appropriate restrictions on the values of the material constants since unconstrained optimization does not, in general, guarantee convexity. It is therefore important to be sure that the optimization process is performed within a range of parameters for which convexity is assured. Indeed, it is worth reiterating that for a large number of material parameters a least-square procedure can lead to problems of non-uniqueness associated with the sensitivity of the material parameters to small changes in the data, as pointed out, for example, in Fung’s book ([34], Section 8.6.1). These types of problems are encountered in numerous studies; see, for example, the recent analysis for (isotropic) rubberlike solids by Ogden et al. [35]. With this background information some recent statements in the literature appear misleading; for example “...Fung-type models allow for straightforward finite element analysis (FEA) implementation, and are thus a good starting point for computational simulations of heart valve biomaterials.” (see the Introduction in [4]).

It is perhaps unrealistic to expect that enough data can be made available from a sufficient number of independent tests in order to determine all the constitutive functions. At present there are certainly not enough data available of the required kind. Indeed, there are not even enough data to enable discrimination between the influences of the invariants $I_4$ and $I_6$ in the transversely isotropic case, let alone the more complex orthotropic case. It is, therefore, necessary to make informed choices about the functional dependence of $W$. And this must be based partly on more detailed information on, for example, the distribution of collagen fibre orientations within the individual soft tissues.

A realistic and relatively simple starting point is, in the case of transverse isotropy, to consider $W$ to depend only on $I_1$ and $I_4$, where $I_4$ is associated with a preferred fibre direction or the mean of a distribution of fibre directions. Models that have been constructed within this framework do not suffer from the deficiencies mentioned above in respect of Fung-type models. Then, the resulting functions $W_1$ and $W_4$ can be fully determined from standard biaxial tests (without superimposed shear). Appropriate predictions can then be made on the basis of the $W$ thereby constructed. If data subsequently obtained are found not to be consistent with such predictions then is the time to refine the model. But, in the absence of such data there is no advantage in considering more complex models. This applies, a fortiori, also to orthotropic materials, where, in the present state of knowledge it suffices to consider $W$ to depend on $I_1, I_4$ and $I_6$. Such a model is the simplest form that accounts for two families of fibres and is adopted because there is not sufficient data available to be able to distinguish between the effects of the different anisotropic invariants.

Finally in this discussion a word of caution is called for. No experiment is perfect. The basic assumptions that underly the experiments are in general satisfied approximately, not exactly, because of experimental errors and the inability to perform the tests strictly in accordance with the assumptions, not least because of the influence of edge effects propagating into the region of measurement.
APPENDIX. A NOTE ON THE CLASSICAL LINEAR THEORY OF ORTHOTROPIC ELASTICITY

In linear elasticity the strain-energy function has the form

\[ W = \frac{1}{2} c_{ijkl} e_{ij} e_{kl}, \tag{43} \]

where \( e_{ij} \) are the components of the infinitesimal strain tensor and \( c_{ijkl} \) are the components of the linear elasticity tensor, which possess the symmetries \( c_{ijkl} = c_{klij} = c_{ijlk} \). From this we calculate the stress components

\[ \sigma_{ij} = \frac{\partial W}{\partial e_{ij}} = c_{ijkl} e_{kl}, \tag{44} \]

noting that, since \( e_{ij} \) is symmetric, \( \partial e_{ij}/\partial e_{kl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 \), where \( \delta_{ij} \) denotes the Kronecker delta.

For an orthotropic material, with the Cartesian axes taken to coincide with the axes of orthotropy, using the Voigt notation for \( c_{ijkl} \), we obtain the reduced forms

\[ W = \frac{1}{2} c_{11} e_{11}^2 + \frac{1}{2} c_{22} e_{22}^2 + \frac{1}{2} c_{33} e_{33}^2 + c_{12} e_{11} e_{22} + c_{13} e_{11} e_{33} + c_{23} e_{22} e_{33} + c_{44} e_{23}^2 + c_{55} e_{13}^2 + c_{66} e_{12}^2, \tag{45} \]

and

\[ \sigma_{11} = c_{11} e_{11} + c_{12} e_{22} + c_{13} e_{33}, \tag{46} \]
\[ \sigma_{22} = c_{12} e_{11} + c_{22} e_{22} + c_{23} e_{33}, \tag{47} \]
\[ \sigma_{33} = c_{13} e_{11} + c_{23} e_{22} + c_{33} e_{33}, \tag{48} \]
\[ \sigma_{12} = 2c_{66} e_{12}, \quad \sigma_{13} = 2c_{55} e_{13}, \quad \sigma_{23} = 2c_{44} e_{23}. \tag{49} \]

We are concerned with the deformation of a thin sheet, in the \((1, 2)\) plane, such that \( e_{13} = e_{23} = 0 \) and hence \( \sigma_{13} = \sigma_{23} = 0 \). Moreover, we adopt the “membrane” approximation \( \sigma_{33} = 0 \), so that

\[ e_{33} = -\frac{c_{13} e_{11}}{c_{33}} - \frac{c_{23} e_{22}}{c_{33}}. \tag{50} \]

Then, on elimination of \( e_{33} \) from (46)–(48), we have

\[ \sigma_{11} = c_1 e_{11} + c_3 e_{22}, \quad \sigma_{22} = c_3 e_{11} + c_2 e_{22}, \quad \sigma_{12} = c_4 e_{12}, \tag{51} \]

where we have defined the notations.
\[ c_1 = c_{11} - \frac{c_{13}^2}{c_{33}}, \quad c_2 = c_{22} - \frac{c_{23}^2}{c_{33}}, \quad c_3 = c_{12} - \frac{c_{13}c_{23}}{c_{33}}, \quad c_4 = 2c_{66}. \] (52)

Equations (51) describe the planar response of a compressible material. For an incompressible material Equations (46)–(48) are replaced by

\[ \sigma_{11} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} - p, \] (53)

\[ \sigma_{22} = c_{12}e_{11} + c_{22}e_{22} + c_{23}e_{33} - p, \] (54)

\[ \sigma_{33} = c_{13}e_{11} + c_{23}e_{22} + c_{33}e_{33} - p, \] (55)

where \( \sigma_{33} = 0 \), and Equation (50) is replaced by the incompressibility condition \( e_{11} + e_{22} + e_{33} = 0 \). After elimination of \( e_{33} \) we again obtain Equations (51), but now with

\[ c_1 = c_{11} + c_{33} - 2c_{13}, \quad c_2 = c_{22} + c_{33} - 2c_{23}, \quad c_3 = c_{33} + c_{12} - c_{13} - c_{23}, \] (56)

instead of \((52)_1\)–\(3\), with \( c_4 \) as in \((52)_4\). It is worth emphasizing that for such (orthotropic) materials the shear and normal behaviors are decoupled.

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REFERENCES


